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Quenched Disorder in a Hierarchical Coulomb Gas Model

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The effects of quenched disorder on the two-dimensional Coulomb gas are studied in the hierarchical approximation. The quenched random variables interact with the charges via a potential that decays as an inverse power (α) of the distance. Recursion relations for the single block charge activities are derived in which the quenched variables explicitly appear. In a linear approximation, for all $\alpha \geqslant 1$, with some restrictions on the variance of the normally distributed random variables, it is shown that the charge activities converge to the Kosterlitz–Thouless fixed point for all sufficiently low temperatures and sufficiently large blocks. The annealed system is also examined. This model is shown to have a Kosterlitz–Thouless phase only for an intermediate range of temperatures. At low temperatures the activities can diverge, and large charges can exist on all length scales.

KEY WORDS: Quenched disorder; Coulomb gas; spin-glasses.

1. INTRODUCTION

One of the most difficult problems of current interest in condensed matter physics is the treatment of "disordered" materials. Much of the interest arises from the behavior of magnetic materials known as spin-glasses which appear to exhibit an equilibrium phase transition of an unusual type.(1–3) A great deal of effort has been expended in developing a variety of models for spin-glass materials,(1,2) of which the mean field Sherrington–Kirkpatrick model is the most notable example,(4) in an effort to elucidate the nature of this phase transition. Only in a few special cases have these models yielded to rigorous analysis.(5–10)

The two-dimensional XY and Villain models with quenched disorder have been studied in the context of spin-glasses.(11–14) Without disorder

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these two-dimensional models undergo a Kosterlitz–Thouless phase transition (at a critical temperature β_{KT}) which can be connected with the phase transition occurring in the two-dimensional Coulomb gas via the sine-Gordon transformation.[15–19,14] The vortices present in the spin vector field can be viewed as two-dimensional electrostatic charges. At high temperatures ($\beta < \beta_{\mathrm{KT}}$) the gas exhibits Debye screening,[20] while at $\beta > \beta_{\mathrm{KT}}$ charges become bound in dipoles and screening no longer occurs.[16,17,21]

The effect of disorder on the behavior of the two-dimensional XY model can be translated into an equivalent problem involving a Coulomb gas with quenched disorder. Previous investigations of both the disordered XY and Villain models have considered two types of quenched disorder: random bond disorder[11,12,14] and a random Dzyaloshinskii–Moriya (DM) interaction.[13] Each of these has a different interpretation in the Coulomb gas picture.

Random bond disorder has been considered by Villain,[14] Fradkin *et al.*,[11] and Jose.[12] The randomness is incorporated into the models as a set of random phase shifts χ_{ij}, so that, in the case of the nearest neighbor XY model, the interaction becomes

$$V_{ij} = \cos(\theta_i - \theta_j - \chi_{ij}) \tag{1}$$

with $\chi_{ij} = -\chi_{ij}$, i and j nearest neighbors on the square lattice Z^2. In the Coulomb gas representation the effect of this type of disorder is to produce charges $q_k + f_k$ at the sites of the dual lattice. Here q_k is an integer, but f_k is a noninteger related to the χ_{ij} through the relation

$$2\pi f_k = \left(\sum_{\langle ij \rangle \in \partial k} \chi_{ij} \right) \operatorname{mod}(2\pi) \tag{2}$$

where ∂k denotes the boundary of the plaquette containing the site k. Jóse studied the case in which the f_k could take on only the values 0 and $\pm 1/2$. In the case of dilute randomness, in which most f_k were 0, he found the power law decay of correlations was preserved, but that for densities of frustrations near 1/2 the two-point function decays exponentially fast for large separations and the magnetic susceptibility is finite at low temperatures. Large amounts of fractional charge disorder destroy the KT phase transition.

The random DM interaction was first considered by Rubinstein *et al.* (RSN)[13] and occurs in addition to the usual XY spin–spin coupling. The form of the random DM interaction is

$$J_{ij} \hat{z} \cdot (\mathbf{s}_i \times \mathbf{s}_j) \tag{3}$$

where the J_{ij} are random variables of mean 0 and variance σ^2, and i and j are nearest neighbors. RSN conclude that this model is equivalent to the two-dimensional Coulomb gas perturbed by a set of quenched random dipoles $\mathbf{p}(\mathbf{r})$. The charge–dipole interaction energy is given, in the continuum approximation, by

$$\sum_{i,j} q_i V_{i,j} = \sum_i q_i \int d\mathbf{r} \, \frac{\mathbf{p}(\mathbf{r}) \cdot (\mathbf{r}_i - \mathbf{r})}{(\mathbf{r}_i - \mathbf{r})^2} \tag{4}$$

The most interesting feature of this model that RSN uncover is that, in addition to the KT transition, there is a new vortex (dipole) unbinding phase transition that occurs at some $\beta_c > \beta_{\mathrm{KT}}$ provided σ is not too great. For $\beta > \beta_c$ activities for nonzero charges begin to grow. All evidence of phase transitions disappears completely for variances larger than a critical value σ_c.

An interesting connection between two-dimensional Coulomb gas models and networks of superconducting Josephson junctions has been made. Teitel and Jayaprakash,[22] in connection with resistive transitions in networks of Josephson junctions in magnetic fields, studied the behavior of uniformly frustrated XY models as the frustration was varied. Granato and Kosterlitz (GK)[23,24] considered the possibility of adding randomness to such networks in the form of random positioning of the network nodes and random variations in the size of the network elements. They showed that in the former case the network could be mapped into the Coulomb gas with quenched random dipoles, while the latter mapped into the Coulomb gas with fractional charges. Based on the analysis of RSN, GK deduce that a disorder-induced low-temperature phase transition may occur in these networks in the case in which all f_k are integers.

Choi *et al.*[25] investigated networks similar to those of GK, but in which all $f_k = 1/2$ (the fully frustrated limit). They find no evidence for reentrant behavior in this model, although their results are not conclusive.

These Josephson junction networks provide a concrete experimental setting in which the predicted behavior of these models can be tested. Experiments and numerical studies[26–29] have been undertaken in this direction; however, at present there seems no strong evidence for the occurrence of this low-temperature vortex unbinding transition.

In this work I consider the effects of adding quenched disorder to the hierarchical Coulomb gas model of Marchetti and Perez (MP).[30,31] In the next section the structure of the model and the form in which the quenched disorder is to be introduced will be discussed. Nonlinear equations governing the behavior of the single block charge activities, and involving the quenched random variables, are derived. It is the behavior of the block charge activities which will be studied in this work. In Section 3 iterations of the linearized version of the equations derived in Section 2 are studied. Under a variety of conditions the behavior of the single block charge activities shows no growth at low temperatures, i.e., no sign of a reentrant transition, although the existence of a critical variance cannot be completely ruled out. Finally, in Section 4 the annealed version of this model is studied, and I show that while the KT phase occurs for an intermediate range of temperatures, at low temperatures large charges form an all length scales of the model.

2. THE MODEL

In the two-dimensional lattice Coulomb gas, a set of integer charges $\{q_i\}$ interact through a potential of the form

$$V_{ij} \approx -\frac{1}{2\pi} \ln(d(i, j)) \tag{5}$$

where $i, j \in Z^2$. The partition function for this gas can be written

$$Z = \sum_{\substack{\{q_i\} \\ \sum_i q_i = 0}} \prod_{i \in \Lambda} \lambda(q_i) \exp\left(-\frac{\beta}{2} \sum_{ij} q_i q_j V_{ij} \right) \tag{6}$$

where the $\lambda(q)$ represent the single-site activities of the charges, analogous to the single-site spin measures present in the hierarchical Dyson spin models.

The hierarchical model of MP is defined by replacing the Coulomb potential in Eq. (6) with the potential

$$V_{ij}^{(h)} = -\frac{1}{2\pi} \ln(d_L(i, j)) \tag{7}$$

where the hierarchical distance $d_L(i, j)$ is

$$d_L(i, j) = \min_{N \geq 1} \{L^N \mid [i/L^N] = [j/L^N]\} \tag{8}$$

and $[x]$ = integer part of x. Here L is a fixed number and represents the basic length scale of the model. The energy of a system of charges in volume Λ_{N_0}, of length L^{N_0}, is

$$E_{\Lambda_{N_0}}(\{q\}) = \sum_{i, j \in \Lambda_{N_0}} q_i q_j V_{ij}^{(h)} \tag{9}$$

The introduction of disorder in the form of fractional charges can easily be accomplished by simply replacing $q_i \to q_i + f_i$, and where the overall neutrality condition becomes $\sum_i (q_i + f_i) = 0$. In the two-dimensional model there will generally be correlations between the values of the f_i, but here there is no need to assume this.

The "random dipole interaction" takes the schematic form

$$V_{\text{ran}}(\{q_i\}) = \sum_{i,j \in \Lambda_{N_0}} q_i \frac{\zeta_i}{d_L^\alpha(i, j)} \tag{10}$$

where the ζ_i are independent and identically distributed random variables, and the angular dependence does not appear. However, this preserves both the power law decay and the random variation in magnitude and sign of the original potential. A power $\alpha \geqslant 1$ has been added to the denominator of Eq. (10), allowing for arbitrary power law interactions to be considered.

The statistical mechanics of this model is determined by the partition function

$$Z = \sum_{\substack{\{q_i\} \\ \sum_i q_i + f_i = 0}} \prod_{i \in \Lambda} \lambda(q_i)\, e^{-(\beta/2)\, E_{\Lambda_{N_0}}(\{q+f\}) - (\beta/2)\, V_{\text{ran}}(\{q_i\})} \tag{11}$$

where

$$E_\Lambda(\{q+f\}) = \sum_{i,j \in \Lambda} (q_i + f_i)(q_j + f_j)\, V_{ij}^{(h)} \tag{12}$$

It is now possible to sum out all charges on length scale L and replace them with block charges on length scale L^2. An easy extension of the analysis of MP shows that

$$\begin{aligned} &\sum_{ij} (q_i + f_i)(q_j + f_j)\, V_{ij}^{(h)} \\ &\qquad = \frac{1}{2\pi} \ln(L) \sum_b (q_b + f_b)^2 + \sum_{b,b'} (q_b + f_b)(q_{b'} + f_{b'})\, V_{b,b'}^{(h)} \end{aligned} \tag{13}$$

where each block b consists of the L^2 sites i from the original model for which $[i/L] = m$ for some fixed m, and

$$q_b = \sum_{i \in b} q_i \tag{14}$$

$$f_b = \sum_{i \in b} f_i \tag{15}$$

The random dipole part of the interaction can be written as

$$V_{\text{ran}}(\{q_i\}) = c_L \sum_{i \in \Lambda_N} q_i \zeta_i + \sum_{b,b'} q_b \frac{\zeta_{b'}}{d_L^\alpha(b, b')} \tag{16}$$

where the block random variable ζ_b is

$$\zeta_b = \frac{1}{L^\alpha} \sum_{i \in b} \zeta_i \tag{17}$$

and $c_L = L^{-\alpha}(1 - L^{-\alpha})$.

Equation (17) suggests the following. If the ζ_i are assumed independent and identically distributed normal random variables, if $\alpha > 1$, the variance of the ζ_b is driven to 0 as the length scale becomes larger. This suggests that the disorder may have a local effect on the system, but on sufficiently long length scales may not. If $\alpha = 1$, the variance of the block random variables is unchanged under the scaling and so the disorder may have more of an effect on the system.

With Eqs. (13) and (16) we can write the partition function in terms of the block variables as

$$Z = \sum_{\{q_b\}} \prod_b \lambda_{b, f_b, \zeta_b}(q_b)\, e^{-(\beta/2)\, E_{\Lambda_{N-1}}(\{q_b + f_b\}) - (\beta/2)\, V_{\text{ran}}(\{q_b\})} \tag{18}$$

where the neutrality condition $\sum_b (q_b + f_b) = 0$ holds. The new single block charge activities λ_{b, f_b} are given by the equation

$$\lambda_{b, f_b, \zeta_b}(q_b) = e^{-(\beta/4\pi)\ln(L)(q_b + f_b)^2} \sum_{\substack{\{q_i\}_{i \in b} \\ \sum q_i = q_b}} \prod_{i \in b} \lambda(q_i)\, e^{-(\beta/2)\, c_L q \zeta_i} \tag{19}$$

Defining the functions

$$\bar{\lambda}_i(q_i) = \lambda(q)\, e^{-(\beta/2)\, c_L q \zeta_i}$$

we can write this more compactly as

$$\begin{aligned} \bar{\lambda}_{b, f_b, \zeta_b}(q_b) = {} & e^{-(\beta/4\pi)\ln(L)(q_b + f_b)^2 - (\beta/2)\, c_L q_b \zeta_b} \\ & \times (\bar{\lambda}_{1,\zeta_1} * \bar{\lambda}_{2,\zeta_2} * \cdots * \bar{\lambda}_{L^2,\zeta_{L^2}})(q_b) \end{aligned} \tag{20}$$

where the operation $*$ denotes convolution.

To simplify the analysis, all f_i are set equal to zero, leaving only the schematic random DM interaction. With $L^2 = 2$, Eq. (20) becomes

$$\bar{\lambda}_{b, \zeta_b}(q_b) = e^{-(\beta/4\pi)\ln(L)\, q_b^2 - (\beta/2)\, c_L q_b \zeta_b} (\bar{\lambda}_{1,\zeta_1} * \bar{\lambda}_{2,\zeta_2})(q_b) \tag{21}$$

Following MP, the equations for the activities can be normalized, and written in the form $\lambda(q) = \delta_{q=0} + \varepsilon(q)$, with $\varepsilon(0) = 0$. Then the single block charge activities satisfy

$$\varepsilon_{b, \zeta_b}(q_b) = e^{-(\beta/4\pi)\ln(L)\, q_b^2 - (\beta/2)\, c_L q_b \zeta_b} \frac{(\varepsilon_1 + \varepsilon_2)(q_b) + (\varepsilon_1 * \varepsilon_2)(q_b)}{1 + (\varepsilon_1 * \varepsilon_2)(0)} \tag{22}$$

for $q_b \neq 0$.

For each site i, take the function

$$\varepsilon_i(q_i) = \varepsilon(q_i)\, e^{-(\beta/2)\, c_L q_i \zeta_i} \tag{23}$$

to be an element of l_1, the Banach space of absolutely summable sequences with norm $\|f\|_1 = \sum_q |f(q)|$.

Define

$$T(\varepsilon_1, \varepsilon_2)(q) = \frac{(\varepsilon_1 + \varepsilon_2)(q) + (\varepsilon_1 * \varepsilon_2)(q)}{1 + (\varepsilon_1 * \varepsilon_2)(0)} \tag{24}$$

and also, for any $f \in l_1$, define the functions H_{ζ_b} by

$$(H_{\zeta_b} f)(q) = H_{\zeta_b}(q)\, f(q) \tag{25}$$

$$= e^{-(\beta/4\pi)\ln(L)\, q^2 - (\beta/2)\, c_L \zeta_b q} f(q) \tag{26}$$

It is fairly simple to show that if $\varepsilon_i \in l_1$ for all i, then $H_{\zeta_b} T\colon l_1 \times l_1 \to l_1$.

Finally, if we consider the subspace of l_1

$$\tilde{l}_1 = \{\varepsilon \in l_1 \mid \varepsilon(0) = 0\} \tag{27}$$

then for each block b,

$$H_{\zeta_b} T\colon\ \tilde{l}_1 \times \tilde{l}_1 \to \tilde{l}_1 \tag{28}$$

Let B denote the collection of all subblocks b of the system. With the ordering $\subseteq$, the set B forms a directed system.[32] The collection of all functions $\{\varepsilon_b\}_{b \in B}$ indexed by the elements of B forms a net in l_1, or more specifically in $\tilde{l}_1$. If I is a directed system with ordering $\succ$, then there is the following notion of convergence[32]:

Definition 1. A net $\{x_\alpha\}_{\alpha \in I}$ in a topological space S is said to converge to a point $x \in S$ if for any neighborhood N of x, there is a $\beta \in I$ so that $x_\alpha \in N$ if $\alpha \succ \beta$.

The presence of a KT phase should appear as the convergence of the net of nonzero charge activities to 0, leaving only a unit activity concentrated on the charge $q = 0$. The existence of a reentrant phase transition, in which free charges again begin to appear, should be characterized by a net of charge activities which exhibits growth in the direction of at least some nonzero values of q.

Note that if $\varepsilon = 0$ for each site i, then

$$H_{\zeta_b} T(0, 0) = 0 \tag{29}$$

Motivated by linear stability theorems for conventional fixed-point equations, consider replacing the map T with its Frechet derivative

$$A(\varepsilon_1, \varepsilon_2) = \varepsilon_1 + \varepsilon_2 \tag{30}$$

The equation governing the behavior of the charge activities is then

$$\varepsilon_b(q_b) = [H_{\zeta_b} A(\varepsilon_1, \varepsilon_2)](q_b) \tag{31}$$

In the next section we consider the behavior of the net $\{\varepsilon_b\}_b$ determined by this family of equations.

3. BEHAVIOR OF THE CHARGE ACTIVITIES

In this section we label all activities $\varepsilon_b^{(n)}$, where n indicates b is a block on length scale L^{n+1}. The size of the system is L^{N_0}, implying $0 \leqslant n \leqslant N_0$. The limit of large n implies allowing N_0 to simultaneously become large, but because the behavior of the block charge activity depends only on quantities within the block, this will not be of concern here.

3.1. Weak Disorder, $\alpha \geqslant 2$

We begin with a simple case, in which the variables $\{\zeta_i\}$ are independent and identically distributed, with distribution having support on the compact interval $[-W, W]$.

Theorem 1. Let $0 < \delta < 1$. If $\alpha \geqslant 2$, then there exists an n_0, depending on δ, such that if $n > n_0$, then for $\beta > 8\pi/(1-\delta)$ there exists a $0 < \kappa = \kappa(\beta, \delta) < 1$ such that

$$\|\varepsilon_b^{(n)}\|_1 \leqslant \kappa^n \, \|\varepsilon^{(0)}\|_1 \tag{32}$$

for any block b on length scale L^n.

From Eqs. (31) and (17) it follows that for any block b,

$$\varepsilon_b^{(1)}(q) \leqslant 2e^{-(\beta/4\pi)\ln(L)\, q^2 + \beta c_L |q| \, W(1 + (2/L^\alpha))} \varepsilon^{(0)}(q) \tag{33}$$

where the random variables have all been replaced by their maximum values. Iterating this procedure yields

$$\varepsilon_b^{(1)}(q) \leqslant 2^n e^{-(n\beta/4\pi)\ln(L)\, q^2 + \beta W c\, |q|} \varepsilon^{(0)}(q) \tag{34}$$

where $c = c_L \sum_{k=0}^{n} L^{k(2-\alpha)}$. If

$$n_0 = [4Wc\pi/\delta \ln(L)] + 1 \tag{35}$$

and $n > n_0$, then for all charges q

$$\varepsilon_b^{(n)}(q) \leqslant 2^n e^{-(\beta/4\pi)\ln(L)(1-\delta)} \varepsilon^{(0)}(q) \tag{36}$$

From this last the following is clear:

$$\|\varepsilon_b^{(n)}\|_1 \leqslant 2^n e^{-(\beta n/4\pi)\ln(L)(1-\delta)} \|\varepsilon^{(0)}\|_1 \tag{37}$$

With $\kappa = 2e^{-(\beta/4\pi)\ln(L)(1-\delta)}$ the result follows. ∎

3.2. Gaussian Disorder

In order to deal with normally distributed random variables, it is convenient to iterate Eq. (31) and rewrite the result. For a block b at the level $n = 1$

$$\varepsilon_{b,\zeta_b}^{(1)}(q) = h_{\beta,L}(q)(\varepsilon_{\zeta_i}^{(0)}(q)\, e^{-(\beta/2)\, c_L q \zeta_b} + \varepsilon_{\zeta_j}^{(0)}(q)\, e^{-(\beta/2)\, c_L q \zeta_b}) \tag{38}$$

where i and j are the two (L^2) sites in block b, and

$$h_{\beta,L}(q) = e^{-(\beta/4\pi)\ln(L)\, q^2} \tag{39}$$

Recalling Eq. (23),

$$\varepsilon_b^{(1)}(q) = h_{\beta,L}(q)(\varepsilon^{(0)}(q)\, e^{-(\beta/2)\, c_L q(\zeta_i + \zeta_b)} + \varepsilon^{(0)}(q)\, e^{-(\beta/2)\, c_L q(\zeta_j + \zeta_b)}) \tag{40}$$

Using Eq. (17) yields

$$\begin{aligned}\frac{\varepsilon^{(1)}(q)}{\varepsilon^{(0)}(q)} = h_{\beta,L}(q)\Bigg(&\exp\left\{-\frac{\beta}{2} c_L q\left(\zeta_i\left(1 + \frac{1}{L^\alpha}\right) + \zeta_j\left(\frac{1}{L^\alpha}\right)\right)\right\} \\ &+ \exp\left\{-\frac{\beta}{2} q c_L\left(\zeta_j\left(1 + \frac{1}{L^\alpha}\right) + \zeta_i\left(\frac{1}{L^\alpha}\right)\right)\right\}\Bigg)\end{aligned} \tag{41}$$

Repeating this procedure n times, we find

$$\frac{\varepsilon_b^{(n)}(q)}{\varepsilon^{(0)}(q)} = h_{\beta,L}^n(q) \sum_\gamma \exp\{-\beta c_L q(\boldsymbol{\zeta}_\gamma \cdot \mathbf{c})\} \tag{42}$$

Each γ represents a set of subblocks of a block on scale L^n, taken in a particular order, and can be written as

$$\gamma = \{i, b_1(i), b_2(i), \ldots, b_{n-1}(i)\} \tag{43}$$

where i is a site in the original lattice, and $b_k(i)$ is the set of all sites which are at a distance L^{k+1} from site i. There are $(L^2)^n$ different γ, one for each site in the block on length scale L^n containing the site i. The components of the vector ζ_γ, with γ having initial site i, are given by

$$(\zeta_\gamma)_k = \begin{cases} \zeta_i, & k=1 \\ \left(\dfrac{1}{L^{k-2}}\right) \sum\limits_{j\in b_k(i)} \zeta_j, & 1<k\leqslant n \end{cases} \tag{44}$$

where $b_k(i)\in\gamma$. While the random variables appearing as components in the ζ_γ are independent, the components of ζ_γ and $\zeta_{\gamma'}$ are not.

The vector $\mathbf{c}$ is the vector with n components given by

$$c_k = \begin{cases} \sum\limits_{j=0}^{n} L^{-\alpha j}, & k=1 \\ L^{-(k-1)(\alpha-1)-1} \sum\limits_{j=0}^{n-k} L^{-\alpha j}, & 1<k\leqslant n \end{cases}$$

The c_k form a decreasing sequence with

$$\frac{1}{L^{(k-1)(\alpha-1)-1}} < c_k < \frac{1}{L^{(k-1)(\alpha-1)-1}} \frac{L^\alpha}{L^\alpha - 1} \tag{45}$$

Since the $\{\zeta_i\}$ are taken to be independent, normally distributed random variables, with mean 0 and variance σ, it is convenient to incorporate the variance directly into Eq. (42) so that all ζ_i are of unit variance.

$$\frac{\varepsilon_b^{(n)}(q)}{\varepsilon^{(0)}(q)} = h^n_{\beta,L}(q) \sum_\gamma \exp\{-\beta\sigma c_L q(\zeta_\gamma\cdot\mathbf{c})\} \tag{46}$$

From Eq. (44) it is clear that each of the components of ζ_γ will also be normally distributed with unit variance.

Equation (42) is similar to the partition function of Derrida's generalized random energy model (GREM).[33] Using the probability space consisting of, for all $n>0$, 2^n independent random variables along with the product measure, we can adopt the methods used by Capocaccia *et al.* (CCP)[34] to study the GREM and prove the following result.

Theorem 2. Let $\kappa>1$ and $0<\gamma<1$. If $\alpha>1$, then with probability one there exists an $n_0(\zeta,\gamma)$ and a $0<\delta(\beta)<1$ such that for all $n>n_0(\zeta,\gamma)$,

$$\|\varepsilon^{(n)}\|_1 \leqslant 2n^\kappa \delta^n(\beta)\, \|\varepsilon^{(0)}\|_1 \tag{47}$$

for all σ, and for $\beta>8\pi/(1-\gamma)$.

If $\alpha = 1$, then with probability one there exists an $n_0(\zeta)$ and a $0 < \delta_1(\beta) < 1$ such that for all $n > n_0(\zeta)$,

$$\|\varepsilon^{(n)}\|_1 \leqslant 2n^{\kappa}\delta_1^n \, \|\varepsilon^{(0)}\|_1 \tag{48}$$

provided σ is sufficiently small and β sufficiently large.

We begin by defining the following events for a block of size L^n:

$$A_k(n) = \left\{ \{\zeta_i\} \mid \forall \gamma, \, |\gamma| = k, \, (\zeta_\gamma)^2 \leqslant \left(\sum_{j=1}^{k} 2^{4-j} \right) \ln(2) n \right\}$$

where $|\gamma|$ is the number of blocks in the set γ. Note that the $A_k(n)$ are, for each k, compact, convex subsets of R^n. The indicator functions of these events are given by

$$1_{A_k(n)} = \begin{cases} 1 & \text{if } A_k(n) \text{ occurs} \\ 0 & \text{if not} \end{cases}$$

The following are obtained from the work of CCP[34] with slight modifications which are not presented.

Lemma 1 (CCP). There exists an $n_1(\zeta)$ such that if $n > n_1(\zeta)$, then the following holds with probability one:

$$\prod_{k=1}^{n} 1_{A_k(n)} = 1$$

Lemma 2 (CCP). Let $\kappa > 1$. There exists an $n_2(\zeta)$ such that if $n > n_2(\zeta)$, then the following holds with probability one:

$$\sum_{\gamma} \prod_{k} 1_{A_k(n)} \exp\{-\beta c_L q(\zeta_\gamma \cdot \mathbf{c})\} \leqslant n^{\kappa} E\left(\sum_{\gamma} \prod_{k} 1_{A_k(n)} \exp\{-\beta c_L q(\zeta_\gamma \cdot \mathbf{c})\} \right) \tag{49}$$

Hence for $n > \max\{n_1, n_2\}$ we have

$$\frac{\varepsilon_b^{(n)}(q)}{\varepsilon^{(0)}(q)} \leqslant n^{\kappa} h_{\beta,L}^n(q) \, E\left(\sum_{\gamma} \prod_{k} 1_{A_k(n)} \exp\{-\beta c_L \sigma q(\zeta_\gamma \cdot \mathbf{c})\} \right) \tag{50}$$

with probability one.

Defining the vectors $\mathbf{y}$ and $\tilde{\mathbf{y}}$ in R^n by

$$\mathbf{y} = \zeta/\sqrt{n} \tag{51}$$

$$\mathbf{y} = -\beta c_L \sigma q \mathbf{c} \tag{52}$$

we can write the expectation in Eq. (50) as

$$E\left(\sum_{\gamma}\prod_{k} 1_{A_k(n)} \exp\{-\beta c_L \sigma q(\zeta_\gamma \cdot \mathbf{c})\}\right)$$
$$= 2^n \left(\exp \frac{\tilde{\mathbf{y}}^2}{2}\right)\left(\frac{n}{2\pi}\right)^{-n/2} \int d\mathbf{y}$$
$$\times \exp\left\{-\frac{n}{2}\left(\tilde{\mathbf{y}} - \frac{\tilde{\mathbf{y}}}{n^{1/2}}\right)^2\right\} 1_{A_n} \tag{53}$$

where

$$A_n = \left\{\mathbf{y} \in R^n \mid \forall k \leqslant n, \sum_i^k y_i^2 \leqslant \ln(2) \sum_i^k (2^{4-i})\right\}$$

is a compact, convex subset of R^n.

Consider now the case $\alpha > 1$. Then an easy calculation shows that there exists a constant $c_2(\alpha)$ such that

$$(\tilde{\mathbf{y}})^2 < (\beta \sigma c_L q c_2(\alpha))^2 \tag{54}$$

from which it follows that

$$\mathbf{y} \cdot \tilde{\mathbf{y}} \leqslant \|\mathbf{y}\|\ \|\tilde{\mathbf{y}}\| \tag{55}$$
$$\leqslant c_3(\alpha)\, \beta\sigma\, |q| \tag{56}$$

for $\mathbf{y}$ restricted to the set A_n. In this last the restriction given by 1_{A_n} has been used to replace $\|\mathbf{y}\|$ by its largest possible value. With this, Eq. (53) becomes

$$E\left(\sum_{\gamma}\prod_{k} 1_{A_k(n)} \exp\{-\beta c_L \sigma q(\zeta_\gamma \cdot \mathbf{c})\}\right)$$
$$\leqslant 2^n \exp\{c_3(\alpha)\, n^{1/2} \beta\sigma\, |q|\} \tag{57}$$

Using Eq. (57) in Eq. (50), we obtain

$$\varepsilon_b^{(n)}(q) \leqslant 2^{n-1} n^{\kappa} \left(\exp\left\{-\frac{\beta n}{4\pi} \ln(L)\, q^2 + c_3(\alpha)\, n^{1/2} \beta\sigma\, |q|\right\}\right) \varepsilon^{(0)}(q) \tag{58}$$

Summing over all q yields

$$\|\varepsilon^{(n)}\|_1 \leqslant n^{\kappa}\{\exp(2n \ln L)\}$$
$$\times \left(\sum_{q \geqslant 1} \exp\left\{-\beta n\left[\frac{1}{4\pi} \ln(L)\, q^2 - \frac{c_3(\alpha)\sigma}{\sqrt{n}}\, |q|\right]\right\}\right) \|\varepsilon^{(0)}\|_1 \tag{59}$$

Let $n > n_3 = 16\pi^2 c_3^2(\alpha)\sigma^2/\gamma^2 \ln^2 L$, with $0 < \gamma < 1$; then

$$\|\varepsilon^{(n)}\|_1 \leqslant n^\kappa e^{2n \ln L}\left(\sum_{q \geqslant 1} e^{-(\beta n/4\pi)(1-\gamma)\ln(L)q}\right)\|\varepsilon^{(0)}\|_1 \tag{60}$$

$$\leqslant e^{2n \ln L} n^\kappa e^{-(\beta n/4\pi)(1-\gamma)\ln L} \times (1 - e^{-(\beta n/4\pi)(1-\gamma)\ln L})^{-1} \|\varepsilon^{(0)}\|_1 \tag{61}$$

If $\beta > 8\pi/(1-\gamma)$, this last becomes

$$\|\varepsilon^{(n)}\|_1 \leqslant 2n^\kappa \delta(\beta)^n \|\varepsilon^{(0)}\|_1 \tag{62}$$

with

$$\delta(\beta) = e^{[-(\beta/4\pi)(1-\gamma)+2]\ln(L)n} \tag{63}$$

$$< 1 \qquad \text{for} \quad \beta > 8\pi/(1-\gamma) \tag{64}$$

Let n_4 be the value of n for which $2n^\kappa\delta(\beta)^n$ is first less than one. Choosing $n_0 = \max\{n_1, n_2, n_3, n_4\}$ completes the first part of the theorem.

In the case $\alpha = 1$ the argument is slightly different. Let $\mathbf{y}^*$ be the vector in A_n whose tip lies at the point in A_n at minimal distance from the point determined by $\tilde{\mathbf{y}}' = \tilde{\mathbf{y}}/\sqrt{n}$. As shown by CCP, then,

$$(\mathbf{y} - \mathbf{y}')^2 \geqslant (\mathbf{y} - \mathbf{y}^*)^2 + (\mathbf{y}^* - \tilde{\mathbf{y}}')^2 \tag{65}$$

from which Eq. (53) becomes

$$E\left(\sum_\gamma \prod_k 1_{A_k(n)} \exp\{-\beta c_L \sigma q(\zeta_\gamma \cdot \mathbf{c})\}\right) \leqslant 2^n \exp\left\{\frac{n}{2}[(\tilde{\mathbf{y}}')^2 - (\mathbf{y}^* - \tilde{\mathbf{y}}')^2]\right\} \tag{66}$$

since the normalized integrals are less than 1.

Since $\alpha = 1$, there exist constants c_1 and c_2 such that

$$\beta^2\sigma^2 c_L^2 q^2 c_1^2 \leqslant (\tilde{\mathbf{y}}')^2 \leqslant \beta^2\sigma^2 c_L^2 q^2 c_2^2 \tag{67}$$

which follows from the definitions of $\mathbf{c}$ and $\tilde{\mathbf{y}}$. If the vector $\tilde{\mathbf{y}}'$ lies in the region A_n, then $\mathbf{y}^* = \tilde{\mathbf{y}}'$ and

$$\frac{\varepsilon^{(n)}(q)}{\varepsilon^{(0)}(q)} \leqslant n^\kappa \exp\left\{n\left(2\ln(L) - \frac{\beta}{4\pi}\ln(L)\,q^2 + \frac{(\tilde{\mathbf{y}}')^2}{2}\right)\right\} \tag{68}$$

Note that for this bound there is a range of temperatures $(\beta_-(q), \beta_+(q))$ for which the exponent is negative, with $\beta_-(1) \to 8\pi$ as $\sigma \to 0$. This range is centered on the value $1/8\pi\sigma^2 c_L^2 c_2^2$ for each q.

For $\beta > \beta_+(1)$ this last bound is of no use. However, the estimate is slightly different if $\tilde{\mathbf{y}}'$ is not in the region A_n. A sufficient condition for the vector $\tilde{\mathbf{y}}'$ to lie outside the region A_n is that

$$(\tilde{\mathbf{y}}')^2 \geq 2^4 \ln(2) \sum_{i=1}^{n} 2^{-i} \tag{69}$$

which will certainly be satisfied if

$$2^4 \ln(2) \sum_{i=0}^{n} 2^{-i} < \beta^2 \sigma^2 c_L^2 q^2 c_1^2 \tag{70}$$

This will be true provided $\beta > 4\sqrt{2}/\sigma c_L c_1$, in which case Eq. (46) can be bounded by

$$\varepsilon^{(n)}(q) \leq n^\kappa 2^n \left(\exp\left\{ -\frac{\beta n}{4\pi} \ln(L)\, q^2 - \frac{n}{2} (\mathbf{y}^*)^2 + n\, |\mathbf{y}^*|\, |\tilde{\mathbf{y}}'| \right\} \right) \varepsilon^{(0)}(q) \tag{71}$$

$$\begin{aligned} &\leq n^\kappa 2^n \left(\exp\left\{ -\frac{n}{2} (\mathbf{y}^*)^2 \right\} \right. \\ &\quad \left. \times \exp\left\{ -n\beta \left(\frac{1}{4\pi} \ln(L)\, q^2 - \sigma c_L c_2\, |q| \right) \right\} \right) \varepsilon^{(0)}(q) \end{aligned} \tag{72}$$

Clearly, if $\sigma < (\ln L)/4\pi c_L c_2$, then the exponent

$$\frac{1}{4\pi} \ln(L)\, q^2 - \sigma c_L c_2\, |q|$$

is negative for all q, yielding

$$\|\varepsilon^{(n)}\|_1 \leq 2n^\kappa \delta_1^n(\beta)\, \|\varepsilon^{(0)}\|_1 \tag{73}$$

with $\ln \delta_1 = 2\ln(L) - \beta[(\ln L)/4\pi - \sigma c_L c_2]$.

Requiring that

$$\sigma < c_1/32\sqrt{2}\, \pi c_L c_2^2 \tag{74}$$

guarantees that for all q the value of β at which $\tilde{\mathbf{y}}'$ leaves A_n occurs prior to $\beta_+(q)$. This establishes the result. ∎

4. MODEL WITH ANNEALING

Annealing corresponds to the situation in which the disorder is in thermal equilibrium with the rest of the system. Formally, the annealed partition function is given by

$$Z_{\text{ann}} = E_\zeta(Z_\zeta) \tag{75}$$

The behavior of the annealed Coulomb gas can be studied using this partition function.

Explicitly computing the average in Eq. (75), we can deduce the equation satisfied by the charge activities as the model is scaled from length L^{n-1} to length L^n to be

$$\bar{\lambda}^{(n)}(q_b) = e^{[-(\beta/4\pi)\ln(L) + (\beta^2\sigma^2/32\ln L)\, k(L,n)]\, q_b^2}(\bar{\lambda}^{(n-1)} * \bar{\lambda}^{(n-1)})(q_b) \tag{76}$$

where

$$\bar{\lambda}^{(0)}(q) = \lambda^{(0)}(q)\, e^{\beta^2\sigma^2 q^2 c_0/2L}$$

with $\lambda^{(0)}$ the initial single-site charge activity of the model, and

$$k(L, n) = \ln(L) + (c_n/L^n)$$

with c_n an increasing sequence of negative numbers with $0 < |c_n| < 1$. If the activities $\bar{\lambda}^{(0)}$ satisfy $\bar{\lambda}^{(0)}(q) = \bar{\lambda}^{(0)}(-q)$, this property will be preserved under this transformation.

If we take $\bar{\lambda}^{(0)}$ to be in l_1, then the convolution is in l_1. However, $\bar{\lambda}^{(1)}$ need not be in l_1 because the exponent in Eq. (76) will be positive if β is sufficiently large and there is no way to guarantee the summability of the resulting quantity.

Define $\beta_c = 8(\ln^2 L)/\pi\sigma^2 k(L, 1)$. Then for all $\beta < \beta_c$ the exponent in Eq. (76) is negative, and $\bar{\lambda}^{(n)} \in l_1$ for all n. For this range of temperatures the transformation (76) makes sense.

We can rewrite this equation in terms of the nonzero charge activities, yielding

$$\varepsilon^{(n)}(q) = [H_{\beta,\sigma,n} T(\varepsilon^{(n-1)})](q) \tag{77}$$

where for any $f \in l_1$, $H_{\beta,\sigma,n}(q)$ is given by

$$(H_{\beta,\sigma,n} f)(q) = \left(\exp\left\{\left[-\frac{\beta}{4\pi}\ln(L) + \frac{\beta^2\sigma^2}{32\ln L}\, k(L, n)\right] q_b^2\right\}\right) f(q) \tag{78}$$

and T is defined by

$$T(\varepsilon) = \frac{2\varepsilon + (\varepsilon * \varepsilon)}{1 + (\varepsilon * \varepsilon)(0)} \tag{79}$$

Note from Eq. (77) that HT has a true fixed point at $\varepsilon = 0$. The set of ε's for which this equation is defined is the subspace S of the space l_1,

$$S = \{\varepsilon \in l_1 \mid \varepsilon(0) = 0,\ \varepsilon(q) = \varepsilon(-q)\} \tag{80}$$

and HT: $S \to S$, for $\beta < \beta_c$.

The following result can be obtained from a modification of the arguments of Marchetti and Perez.[31]

Theorem 3. Let $\beta < \beta_c$. Then there exists an interval $[\beta_-, \beta_+]$ with $\beta_+ \leqslant \beta_c$ such that if $\beta \in [\beta_-, \beta_+]$ and σ is sufficiently small, then for all $\varepsilon^{(0)} \in S$ the following holds:

$$\|\varepsilon^{(n)}\|_1 \leqslant \left(\frac{7}{12}\right)^{(n-1)} \|\varepsilon^{(0)}\|_1 \tag{81}$$

Let $\varepsilon^{(0)} \in S$. Then, by Hölder's inequality,

$$\|\varepsilon^{(1)}\|_1 = \|H_{\beta,\sigma} T(\varepsilon^{(0)})\|_1 \tag{82}$$

$$\leqslant \frac{2\ \|H_{\beta,\sigma}\varepsilon^{(0)}\|_1 + \|H_{\beta,\sigma}(\varepsilon^{(0)} * \varepsilon^{(0)})\|_1}{1 + \|\varepsilon^{(0)}\|_2^2} \tag{83}$$

where the charge symmetry of $\varepsilon^{(0)}$ has been used to write

$$(\varepsilon^{(0)} * \varepsilon^{(0)})(0) = \sum_q (\varepsilon^{(0)}(q))^2 \tag{84}$$

Now, using Hölder's inequality on the firs term of Eq. (83)

$$2\ \|H_{\beta,\sigma}\varepsilon^{(0)}\|_1 \leqslant 2\ \|H_{\beta,\sigma}\delta_{q \neq 0}\|_2\ \|\varepsilon^{(0)}\|_2 \tag{85}$$

The explicit insertion of the $\delta_{q \neq 0}$ occurs because ε vanishes for charge 0, suppressing the charge-0 value of $H_{\beta,\sigma}$.

The second term in the numerator yields

$$\|H_{\beta,\sigma}(\varepsilon^{(0)} * \varepsilon^{(0)})\|_1 \leqslant \|H_{\beta,\sigma}\delta_{q \neq 0}\|_1\ \|\varepsilon^{(0)} * \varepsilon^{(0)}\|_\infty \tag{86}$$

$$\leqslant \|H_{\beta,\sigma}\delta_{q \neq 0}\|_1 \sup_{q \neq 0}\ [(\varepsilon^{(0)} * \varepsilon^{(0)})(q)] \tag{87}$$

$$\leqslant \|H_{\beta,\sigma}\delta_{q \neq 0}\|_1\ \|\varepsilon^{(0)}\|_2^2 \tag{88}$$

where Hölder's inequality has been used. So,

$$\|\varepsilon^{(1)}\|_1 \leqslant \frac{2\ \|H_{\beta,\sigma}\delta_{q \neq 0}\|_2\ \|\varepsilon^{(0)}\|_2 + \|H_{\beta,\sigma}\delta_{q \neq 0}\|_1\ \|\varepsilon^{(0)}\|_2^2}{1 + \|\varepsilon^{(0)}\|_2^2} \tag{89}$$

Now, when $\beta < \beta_c$,

$$\|H_{\beta,\sigma}\delta_{q \neq 0}\|_2^2 \leqslant \|H_{\beta,\sigma}\delta_{q \neq 0}\|_1^2 \tag{90}$$

since the exponent in $H_{\beta,\sigma}(q)$ is negative and the sums are finite. Equation (89) may be written

$$\|\varepsilon^{(1)}\|_1 \leqslant \|H_{\beta,\sigma}\delta_{q\neq 0}\|_1 \frac{2\|\varepsilon^{(0)}\|_2 + \|\varepsilon^{(0)}\|_2^2}{1+\|\varepsilon^{(0)}\|_2^2} \tag{91}$$

$$\leqslant \|H_{\beta,\sigma}\delta_{q\neq 0}\|_1 \sup_{x\geqslant 0} \frac{2x+x^2}{1+x^2} \tag{92}$$

$$\leqslant 2\|H_{\beta,\sigma}\delta_{q\neq 0}\|_1 \tag{93}$$

Using this estimate of $\varepsilon^{(0)}$, we find that

$$\|\varepsilon^{(1)}\|_1 \leqslant (2\|H_{\beta,\sigma}\|_\infty + 2\|H_{\beta,\sigma}\delta_{q\neq 0}\|_1^2)\|\varepsilon^{(0)}\|_1 \tag{94}$$

Estimating the first term in Eq. (94) yields

$$2\|H_{\beta,\sigma}\delta_{q\neq 0}\|_\infty = \sup_{q\neq 0} \exp\left\{\left(\frac{\beta^2\sigma^2}{32\ln L}k(L,1) - \frac{\beta}{4\pi}\ln L\right)q^2 + 2\ln L\right\} \tag{95}$$

$$\leqslant \exp\left\{\frac{\beta^2\sigma^2}{32\ln L}k(L,1) - \frac{\beta}{4\pi}\ln(L) + 2\ln L\right\} \tag{96}$$

for $\beta<\beta_c$. If

$$\frac{\beta^2\sigma^2}{32\ln L}k(L,1) - \frac{\beta}{4\pi}\ln(L) + 2\ln L < -4\ln L \tag{97}$$

then

$$2\|H_{\beta,\sigma}\|_\infty < \tfrac{1}{4} \tag{98}$$

The condition given by Eq. (97) is satisfied if β lies in the interval $(\beta_-^{(1)}, \beta_+^{(1)})$, where

$$\beta_\pm^{(1)} = \tfrac{1}{2}\beta_c\{1\pm[1-(96\pi/\beta_c)]^{1/2}\} \tag{99}$$

Since $\beta_c \propto 1/\sigma^2$ if σ is sufficiently small, both roots will be real and $\beta_+^{(1)}$ will satisfy $\beta_+^{(1)}<\beta_c$.

Now we estimate the second term in Eq. (94),

$$2\|H_{\beta,\sigma}\delta_{q\neq 0}\|_1^2 = 2\left(2\sum_{q>0} e^{[(\beta^2\sigma^2 k(L,2)/2)-(\beta\ln(L)/4\pi)]\,q^2}\right)^2 \tag{100}$$

$$\leqslant 1/3 \tag{101}$$

where, in Eq. (100) use has been made of Eq. (97).

Using the estimates in (98) and (100) with Eq. (94), we obtain the estimate

$$\|\varepsilon^{(1)}\|_1 \leqslant \frac{7}{12} \|\varepsilon^{(0)}\|_1 \tag{102}$$

Repeating this argument establishes that

$$\|\varepsilon^{(n)}\|_1 \leqslant \left(\frac{7}{12}\right)^{n-1} \|\varepsilon^{(0)}\|_1 \tag{103}$$

However, the range of permitted β changes each time because the factor $k(L, n)$ appearing in the equivalent of Eq. (97) depends on n. The intervals $(\beta_-^{(n)}, \beta_+^{(n)})$ will have nonzero intersection provided $\beta_-^{(\infty)} < \beta_+^{(0)}$, which holds provided σ is sufficiently small. ▌

Thus, for at least the nonempty range of temperatures $\beta \in (\beta_-^{(\infty)}, \beta_+^{(1)})$, and for σ small enough, there is a phase in which the charge activities converge to zero on sufficiently long length scales.

The very low-temperature behavior of this model is also of interest. While it is true that the transformation of the charge activities will not necessarily produce summable sequences, for a finite sequence there is no trouble for a finite number of iterations. Choosing the initial activities to be

$$\varepsilon(q) = \begin{cases} 1 & \text{if} \quad q = \pm 1 \\ 0 & \text{otherwise} \end{cases} \tag{104}$$

then

$$T(\varepsilon)(q) = \begin{cases} \frac{2}{3} & \text{if} \quad q = \pm 1 \\ \frac{1}{3} & \text{if} \quad q = \pm 2 \end{cases} \tag{105}$$

and

$$\varepsilon^{(0)}(q) = \begin{cases} \dfrac{2}{3} \exp\left\{ -\dfrac{\beta}{4\pi} \ln(L) + \dfrac{\beta^2 \sigma^2}{2} k(L, 2) \right\} & \text{if} \quad q = \pm 1 \\ \dfrac{1}{3} \exp\left\{ 4\left(-\dfrac{\beta}{4\pi} \ln(L) + \dfrac{\beta^2\sigma^2}{2} k(L, 2) \right) \right\} & \text{if} \quad q = \pm 2 \end{cases} \tag{106}$$

For sufficiently small β the dominant value of the activities occurs for charge ± 2. At each further iteration we double the magnitude of the largest possible charge. For β small enough, $\varepsilon^{(n)}(q)$ takes on its largest value for charges of the largest magnitude. Thus, if $\beta > \beta_c$, the system becomes unstable with respect to the formation of charges, and large charges exist on all length scales. It is clear from this last example that for $\beta > \beta_c$ the free energy of the annealed system will fail to exist because the largest charge appearing in the exponent of the activities will be proportional to the volume.

5. CONCLUSION

In this work I have adapted the hierarchical Coulomb gas model of Marchetti and Perez to include quenched disorder interacting with the charges via a potential that has a power law decay with distance. This potential is intended to approximate the random dipole potential considered previously by RSN. Nonlinear equations describing the evolution of the "random" single block charge activities were derived. These equations were then studied in a "linear" approximation in the hope of finding evidence of a new, second low-temperature phase transition.

In each of the cases considered, no evidence for the new low-temperature phase transition was found. In each case the net of l_1 functions describing the single block activities for nonzero charges converged, with probability one, to the KT fixed point (i.e., $0 \in l_1$), for β sufficiently large, although in the case $\alpha = 1$ the variance was required to be small.

It may be that the nonlinear portion of the transformation of the charge activities cannot be neglected and is crucial to the appearance of the reentrant transition. This portion of the transformation is much more difficult to control, however, and it has not been possible to produce results concerning the full transformation.

An analysis of the annealed model indicates that this model exhibits a KT phase only in an intermediate range of temperatures. At sufficiently low temperatures this model becomes unstable to charge formation with large charges forming on all length scales. It is unclear if this behavior is connected to that found by Rubinstein, Schraiman, and Nelson in their quenched model.

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